

On the Numerical Solution of Nonlinear Fractional-Integro Differential Equations

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Abstract

In the present study, a numerical method, perturbation-iteration algorithm (shortly PIA), have been employed to give approximate solutions of nonlinear fractional-integro differential equations (FIDEs). Comparing with the exact solution, the PIA produces reliable and accurate results for FIDEs.

Keywords: Fractional-integro differential equations, Caputo fractional derivative, Initial value problems, Perturbation-Iteration Algorithm.

1 Introduction

Scientists has been interested in fractional order calculus as long as it has been in classical integer order analysis. However, for many years it could not find practical applications in physical sciences. Recently, fractional calculus has been used in applied mathematics, viscoelasticity [1], control [2], electrochemistry [3], electromagnetic [4].

Developments in symbolic computation capabilities is one of the driving forces behind this rise. Different multidisciplinary problems can be handled with fractional derivatives and integrals.

[5] and [6] are studies that describe the fundamentals of fractional calculus give some applications. Existence and uniqueness of the solutions are also studied in [7].

Similar to the studies in physical sciences, fractional order integro differential equations (FIDEs) also gave scientists the opportunity of describing and modeling many important and useful physical problems.

In this manner, a remarkable effort has been expended to propose numerical methods for solving FIDEs, in recent years. Fractional variational iteration method [8, 9], homotopy analysis method [10, 11], Adomian decomposition method [12, 13] and fractional differential transform method [14–16] are among these methods.

In our study, we use the previously developed method PIA, to obtain approximate solutions of some FIDEs. This method can be applied to a wide range of problems without requiring any special assumptions and restrictions.

A few fractional derivative definitions of an arbitrary order exists in the literature. Two most used of them are the Riemann-Liouville and Caputo fractional derivatives. The two definitions are quite similar but have different order of evaluation of derivation.

The Riemann-Liouville fractional integral of order α is described by:

$$J^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} u(t) dt, \quad \alpha > 0, \quad x > 0. \quad (1)$$

The Riemann-Liouville and Caputo fractional derivatives of an arbitrary order are defined as the following, respectively

$$D^\alpha u(x) = \frac{d^m}{dx^m} (J^{m-\alpha} u(x)) \quad (2)$$

$$D_*^\alpha u(x) = J^{m-\alpha} \left(\frac{d^m}{dx^m} u(x) \right). \quad (3)$$

where $m - 1 < \alpha \leq m$ and $m \in \mathbb{N}$.

Due to the appropriateness of the initial conditions, fractional definition of Caputo is often used in recent years.

Definition 1.1 *The Caputo fractional derivative of a function $u(x)$ is defined as*

$$D_*^\alpha u(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} u^{(m)}(t) dt, & m-1 < \alpha \leq m \\ \frac{d^m u(x)}{dx^m} & \alpha = m \end{cases} \quad (4)$$

for $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$, $x > 0$, $u \in C_{-1}^m$.

Following lemma gives the two main properties of Caputo fractional derivative.

Lemma 1.2 *For $m - 1 < \alpha \leq m$, $u \in C_\mu^m$, $\mu \geq -1$ and $m \in \mathbb{N}$ then*

$$D_*^\alpha J^\alpha u(x) = u(x) \quad (5)$$

and

$$J^\alpha D_*^\alpha u(x) = u(x) - \sum_{k=0}^{m-1} u^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0. \quad (6)$$

After this introductory section, Section 2 is reserved to a brief review of the Perturbation-Iteration Algorithm PIA, in Section 3 some examples are illustrated to show the simplicity and effectiveness of the algorithm. Finally the paper ends with a conclusion in Section 4.

2 Analysis of the PIA

Differential equations are naturally used to describe problems in engineering and other applied sciences. Searching approximate solutions for complicated equations has always attracted attention. Many different methods and frameworks exist for this purpose and perturbation techniques [17–19] are among them. These techniques can be used to find approximate solutions for both ordinary and partial differential equations.

Requirement of a small parameter in the equation that is sometimes artificially inserted is a major limitation of perturbation techniques that renders them valid only in a limited range. Therefore, to overcome the disadvantages come with the perturbation techniques, several methods have been proposed by authors [20–29].

Parallel to these attempts, a perturbation-iteration method has been proposed by Aksoy, Pakdemirli and their co-workers [33–35] previously. A primary effort of producing root finding algorithms for algebraic equations [30–32], finally guided to obtain formulae for differential equations also [33–35]. In the new technique, an iterative algorithm is constructed on the perturbation expansion. The present method has been tested on Bratu-type differential equations [33] and first order differential equations [34] with success. Then the algorithms were applied to nonlinear heat equations also [35]. Finally, the solutions of the Volterra and Fredholm type integral equations [36] and ordinary differential equation systems [37] have been presented by the developed method.

This new algorithm have not been used for any fractional integro differential equations yet. To obtain the approximate solutions of FIDEs, the most basic perturbation-iteration algorithm PIA(1,1) is employed by taking one correction term in the perturbation expansion and correction terms of only first derivatives in the Taylor series expansion. [33–35].

Take the fractional-integro differential equation.

$$F \left(u^{(\alpha)}, u, \int_0^t g(t, s, u(s)) ds, \varepsilon \right) = 0 \quad (7)$$

where $u = u(t)$ and ε is a small parameter. The perturbation expansions with only one correction term is

$$\begin{aligned} u_{n+1} &= u_n + \varepsilon(u_c)_n \\ u'_{n+1} &= u'_n + \varepsilon(u'_c)_n \end{aligned} \quad (8)$$

Replacing Eq.(8) into Eq.(7) and writing in the Taylor series expansion for only first order derivatives gives

$$\begin{aligned} &F \left(u_n^{(\alpha)}, u_n, \int_0^t g(t, s, u_n(s)) ds, 0 \right) \\ &+ F_u \left(u_n^{(\alpha)}, u_n, \int_0^t g(t, s, u_n(s)) ds, 0 \right) \varepsilon(u_c)_n \\ &+ F_{u^{(\alpha)}} \left(u_n^{(\alpha)}, u_n, \int_0^t g(t, s, u_n(s)) ds, 0 \right) \varepsilon(u_c^{(\alpha)})_n \\ &+ F_{\int u} \left(u_n^{(\alpha)}, u_n, \int_0^t g(t, s, u_n(s)) ds, 0 \right) \varepsilon \int (u_c)_n \\ &+ F_\varepsilon \left(u_n^{(\alpha)}, u_n, \int_0^t g(t, s, u_n(s)) ds, 0 \right) \varepsilon = 0 \end{aligned} \quad (9)$$

or

$$(u_c^{(\alpha)})_n \frac{\partial F}{\partial u^{(\alpha)}} + (u_c)_n \frac{\partial F}{\partial u} + \left(\int (u_c)_n \right) \frac{\partial F}{\partial (\int u)} + \frac{\partial F}{\partial \varepsilon} + \frac{F}{\varepsilon} = 0 \quad (10)$$

Here $(.)'$ represents the derivative according to the independent variable and

$$F_\varepsilon = \frac{\partial F}{\partial \varepsilon}, F_u = \frac{\partial F}{\partial u}, F_{u'} = \frac{\partial F}{\partial u'}, \dots \quad (11)$$

The derivatives in the expansion are evaluated at $\varepsilon = 0$. Beginning with an initial function $u_0(t)$, first $(u_c)_0(t)$ is calculated by the help of (10) and then substituted into Eq.(8) to calculate $u_1(t)$. Iteration procedure is continued using (10) and (8) until obtaining a reasonable solution.

3 Applications

Example 3.1 Consider the following nonlinear fractional-integro differential equation [38]:

$$\frac{d^\alpha u(t)}{dt^\alpha} - \int_0^1 ts(u(s))^2 ds = 1 - \frac{t}{4}, \quad t > 0, \quad 0 \leq t < 1, \quad 0 < \alpha \leq 1 \quad (12)$$

with the initial condition $u(0) = 0$ and the known exact solution for $\alpha = 1$ is

$$u(t) = t \quad (13)$$

Before iteration process rewriting Eq.(12) with adding and subtracting $u'(t)$ to the equation gives

$$\varepsilon \frac{d^\alpha u(t)}{dt^\alpha} - u'(t) + \varepsilon u'(t) - \varepsilon \int_0^1 ts(u(s))^2 ds - 1 + \frac{t}{4} = 0 \quad (14)$$

In this case for

$$F(u', u, \varepsilon) = \frac{1}{\Gamma(1-\alpha)} \varepsilon \int_0^t \frac{u'(s)}{(t-s)^\alpha} ds - u'_n(t) + \varepsilon u'_n(t) - \varepsilon \int_0^1 ts(u_n(s))^2 ds - 1 + \frac{t}{4} \quad (15)$$

and the iteration formula

$$u'(t) + \frac{F_u}{F_{u'}} u(t) = -\frac{F_\varepsilon + \frac{F}{\varepsilon}}{F_{u'}} \quad (16)$$

the terms that will be replaced in, are

$$\begin{aligned} F &= u'_n(t) - 1 + \frac{t}{4} \\ F_u &= 0 \\ F_{u'} &= 1 \\ F_\varepsilon &= -u'_n(t) + \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'(s)}{(t-s)^\alpha} ds - \int_0^1 ts(u(s))^2 ds \end{aligned} \quad (17)$$

After substitution the differential equation for this problem, Eq.(10) becomes

$$\frac{\int_0^t (-s+t)^{-\alpha} u'_n(s) ds}{\Gamma(1-\alpha)} + (u'_c(t))_n = \int_0^1 st(u_n(s))^2 ds + \frac{4-t+4(-1+\varepsilon)u'_n(t)}{4\varepsilon} \quad (18)$$

Appropriate to the initial conditions, chosen $u_0(t) = 0$ and, solving Eq.(18) for $n = 0$ gives

$$(u_c(t))_0 = t - \frac{t^2}{8} + C_1 \quad (19)$$

This expression written in

$$u_1 = u_0 + \varepsilon(u_c(t))_0 \quad (20)$$

gives

$$u_1(x, t) = u_0(x, t) + \varepsilon(t - \frac{t^2}{8} + C_1) \quad (21)$$

or

$$u_1(x, t) = \varepsilon(t - \frac{t^2}{8} + C_1) \quad (22)$$

Solving this equation for

$$u_1(0) = 0 \quad (23)$$

we obtain

$$C_1 = 0 \quad (24)$$

For this value and $\varepsilon = 1$ reorganizing $u_1(t)$

$$u_1(t) = t - \frac{t^2}{8} \quad (25)$$

gives the first iteration result. If the iteration procedure is continued in a similar way, we obtain the following iterations.

$$u_2(t) = 2t - \frac{571t^2}{3840} + \frac{t^{2-\alpha}(t+4(-3+\alpha))}{4\Gamma(4-\alpha)} \quad (26)$$

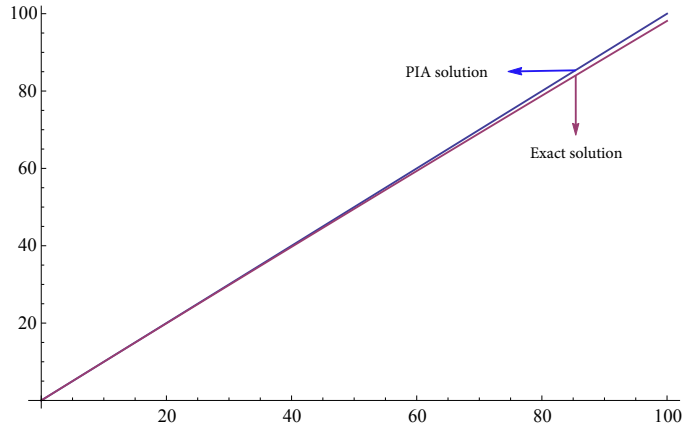


Figure 1: Comparison of the PIA solution $u_3(t)$ and exact solution for Example 3.1. when $\alpha = 1$

$$\begin{aligned}
u_3(t) = & 3t + \frac{29844889t^2}{176947200} - \frac{t^{3-2\alpha}(t + 8(-2 + \alpha))}{4\Gamma(5 - 2\alpha)} \\
& + \frac{t^2(3379230 + 8t^{-\alpha}(1051t + 5760(-3 + \alpha))(-7 + \alpha)(-6 + \alpha)(-5 + \alpha))}{15360(-7 + \alpha)(-6 + \alpha)(-5 + \alpha)\Gamma(4 - \alpha)} \\
& - \frac{2240277\alpha + (450151 - 28436\alpha)\alpha^2}{15360(-7 + \alpha)(-6 + \alpha)(-5 + \alpha)\Gamma(4 - \alpha)} \\
& - \frac{t^2(-4 + \alpha)(-1159 + 2\alpha(529 + 16(-10 + \alpha)\alpha))}{64(-7 + 2\alpha)\Gamma(5 - \alpha)^2}
\end{aligned} \tag{27}$$

The other iterations contain large inputs and are not given. A computational software program could help to calculate the other iterations up to any order. In Table 1. some of the PIA iteration results are compared with the exact solution. The results express that the present method gives highly approximate solutions. Also in Figure 1. the obtained results are illustrated graphically.

Table 1: Numerical results of Example 3.1. for different u values when $\alpha = 1$

$\alpha = 1.0$						
t	u_2	u_3	u_4	u_5	Exact Solution	Absolute Error
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.099763	0.099953	0.099990	0.099981	0.100000	1.872712E-6
0.2	0.199052	0.199812	0.199962	0.199992	0.200000	7.490848E-6
0.3	0.297867	0.299577	0.299915	0.299983	0.300000	1.685440E-5
0.4	0.396208	0.399249	0.399850	0.399970	0.400000	2.996339E-5
0.5	0.494075	0.498826	0.499765	0.499953	0.500000	4.681780E-5
0.6	0.591468	0.598310	0.599662	0.599932	0.600000	6.741763E-5
0.7	0.688388	0.697700	0.699541	0.699908	0.700000	9.176289E-5
0.8	0.784833	0.796996	0.799400	0.799880	0.800000	1.198535E-4
0.9	0.880804	0.896198	0.899241	0.899848	0.900000	1.516896E-4
1.0	0.976302	0.995307	0.999063	0.999812	1.000000	1.872712E-4

Example 3.2 Consider the following system of nonlinear fractional-integro differential equations [39]:

$$\begin{aligned}
\frac{d^{\alpha_1} u(t)}{dt^{\alpha_1}} &= 1 - \frac{1}{2} \left(k'(t) \right)^2 + \int_0^t ((t-s)k(s) + u(s)k(s)) ds \\
\frac{d^{\alpha_2} k(t)}{dt^{\alpha_2}} &= 2t + \int_0^t ((t-s)u(s) - k^2(s) + u^2(s)) ds \quad 0 < \alpha_1, \alpha_2 \leq 1
\end{aligned} \tag{28}$$

Given with $u(0) = 0$, $k(0) = 1$ as initial conditions. The exact solution for $\alpha_1 = \alpha_2 = 1$ is

$$\begin{aligned} u(t) &= \sinh t \\ k(t) &= \cosh t \end{aligned} \quad (29)$$

Rewriting Eq.(28) in the following for with adding and subtracting $u'(t)$ and $k'(t)$ to the equation respectively gives

$$\begin{aligned} \varepsilon \frac{d^{\alpha_1} u(t)}{dt^{\alpha_1}} + u'(t) - \varepsilon u'(t) - 1 + \frac{1}{2}(k'(t))^2 - \varepsilon \int_0^t ((t-s)k(s) - u(s)k(s)) ds \\ \varepsilon \frac{d^{\alpha_2} u(t)}{dt^{\alpha_2}} + k'(t) - \varepsilon k'(t) - 2t - \varepsilon \int_0^t ((t-s)u(s) + k^2(s) - u^2(s)) ds \end{aligned} \quad (30)$$

In this case for

$$\begin{aligned} F(u', u, \varepsilon) &= \frac{1}{\Gamma(1-\alpha_1)} \varepsilon \int_0^t \frac{u'(s)}{(t-s)^{\alpha_1}} ds - \varepsilon \int_0^t ((t-s)k(s) + u(s)k(s)) ds - 1 + \frac{1}{2}(k'(t))^2 \\ F(k', k, \varepsilon) &= \frac{1}{\Gamma(1-\alpha_2)} \varepsilon \int_0^t \frac{u'(s)}{(t-s)^{\alpha_2}} ds - \varepsilon \int_0^t ((t-s)u(s) - k^2(s) + u^2(s)) ds - 2t \end{aligned} \quad (31)$$

and the iteration formula

$$u'(t) + \frac{Fu}{Fu'} u(t) = -\frac{F_\varepsilon + \frac{F}{\varepsilon}}{Fu'} \quad (32)$$

the terms that will be replaced in, are

$$\begin{aligned} F &= u'_n(t) - 1 + \frac{k'_n(t)^2}{2} \\ F_u &= 0 \\ F_{u'} &= 1 \\ F_\varepsilon &= -u'_n(t) + \frac{1}{\Gamma(1-\alpha_1)} \int_0^t \frac{u'_n(s)}{(t-s)^{\alpha_1}} ds - \int_0^t ((t-s)k_n(s) + u_n(s)k_n(s)) ds \end{aligned} \quad (33)$$

and the iteration formula

$$k'(t) + \frac{F_k}{F_{k'}} k(t) = -\frac{F_\varepsilon + \frac{F}{\varepsilon}}{F_{k'}} \quad (34)$$

the terms that will be replaced in, are

$$\begin{aligned} F &= k'_n(t) - 2t \\ F_k &= 0 \\ F_{k'} &= 1 \\ F_\varepsilon &= -k'_n(t) + \frac{1}{\Gamma(1-\alpha_2)} \int_0^t \frac{k'_n(s)}{(t-s)^{\alpha_2}} ds - \int_0^t ((t-s)u_n(s) - k_n(s)^2 + u_n(s)^2) ds \end{aligned} \quad (35)$$

After substitution, the system of differential equations for this problem become

$$\frac{1}{\Gamma(1-\alpha_1)} \int_0^t (-s+t)^{-\alpha_1} u'_n(s) ds + (u'_c(t))_n + \frac{-1 + \frac{1}{2}k'_n(t)^2 + u'_n(t)}{\varepsilon} = \int_0^t k_n(s)(-s+t+u_n(s)) ds + u'_n(t)$$

$$\frac{1}{\Gamma(1-\alpha_2)} \int_0^t (-s+t)^{-\alpha_2} k'_n(s) ds + (k'_c(t))_n = \int_0^t (-k_n(s)^2 + u_n(s)(-s+t+u_n(s))) ds + \frac{2t + (-1+\varepsilon)k'_n(t)}{\varepsilon} \quad (36)$$

Appropriate to the initial conditions, chosen $u_0(t) = 0$ and $k_0(t) = 1$ and solving Eq.(36) for $n = 0, 1, 2, 3, \dots$ the successive iterations are

$$u_1(t) = \frac{1}{6}(6t + t^3) \quad (37)$$

$$k_1(t) = 1 + \frac{t^2}{2} \quad (38)$$

$$u_2(t) = \frac{1}{504}t(1008 + 168t^2 + 21t^4 + t^6) - \frac{t^{2-\alpha_1}(12 + t^2 + (-7 + \alpha_1)\alpha_1)}{\Gamma(5 - \alpha_1)} \quad (39)$$

$$k_2(t) = 1 + t^2 + \frac{t^4}{24} + \frac{t^6}{240} + \frac{t^8}{2016} - \frac{t^{3-\alpha_2}}{\Gamma(4 - \alpha_2)} \quad (40)$$

Following in this manner the third iteration results, $u_3(t)$ and $k_3(t)$, are calculated. Again Table 2, Figure 2 and Figure 3 prove that PIA give remarkably approximate results. We can say that the higher iterations would give closer results.

Table 2: Numerical results of Example 3.2. for u_3 and k_3 values when $\alpha_1 = \alpha_2 = 1$

$\alpha_1 = \alpha_2 = 1$						
t	PIA (u_3)	Exact Solution	Absolute Error	PIA (k_3)	Exact Solution	Absolute Error
0.0	0.000000	0.000000	0.000000	1.000000	1.000000	0.000000
0.1	0.100166	0.100166	1.591577E-10	1.005004	1.005004	1.191735E-11
0.2	0.201335	0.201336	2.053723E-8	1.020066	1.020066	3.060393E-9
0.3	0.304519	0.304520	3.556439E-7	1.045338	1.045338	7.884730E-8
0.4	0.410749	0.410752	2.714842E-6	1.081073	1.081072	7.934216E-7
0.5	0.521082	0.521095	1.326132E-5	1.127630	1.127625	4.774578E-6
0.6	0.636604	0.636653	4.893639E-5	1.185485	1.185465	2.077300E-5
0.7	0.758434	0.758583	1.490491E-4	1.255241	1.255169	7.230620E-5
0.8	0.887710	0.888105	3.950285E-4	1.337648	1.337434	2.139083E-4
0.9	0.025574	0.026516	9.426045E-4	1.433645	1.433086	5.592545E-4
1.0	0.173128	0.175201	2.072716E-3	1.544407	1.543080	1.327116E-3

4 Conclusion

In this study, Perturbation-Iteration Algorithm was introduced for some Fractional Differential Equations. It is clear that the method is very simple and reliable perturbation-iteration technique and producing very approximate results. We expect that the present method could used to calculate the approximate solutions of other types of fractional differential equations.

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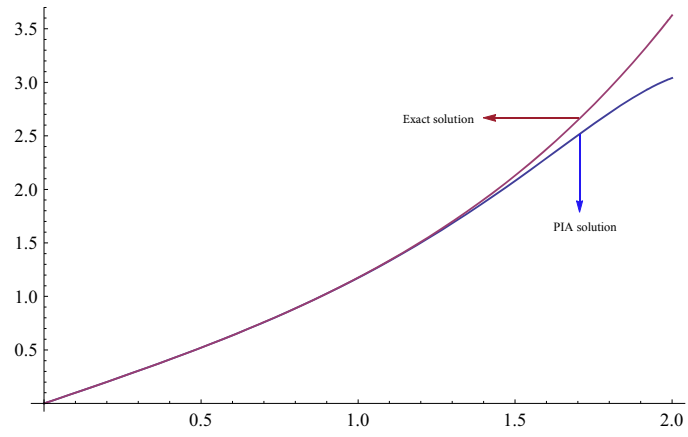


Figure 2: Comparison of the PIA solution ($u_3(t)$) and exact solution for Example 3.2. when $\alpha_1 = \alpha_2 = 1$

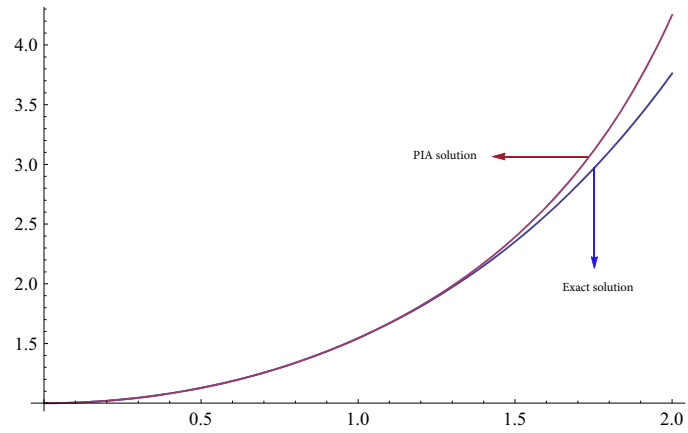


Figure 3: Comparison of the PIA solution ($k_3(t)$) and exact solution for Example 3.2. when $\alpha_1 = \alpha_2 = 1$

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